JGP - Vol. 5, n. 2, 1988

Rational billiards and algebraic curves

E. AURELL

Institute of Theoretical Physics
Chalmers University of Technology and University of GOTEBORG
41296 GOTEBORG, SWEDEN

C. ITZYKSON

Service de Physique Théorique(*) de Saclay
91191 Gif-sur Yvette Cedex, FRANCE

It is a great honour to dedicate this note
to Professor I.M. Gelfand on the occasion
of his seventy fifth birthday

Abstract. *We associate an algebraic curve to a rational triangular billiard and study some of its properties.*

1. Richens and Berry [1] have coined the word pseudo-integrable for those Hamiltonian systems with N degrees of freedom such that phase space is foliated by N-dimensional compact manifolds which are not topologically tori. The simplest example is given by the free motion of a particle elastically reflected by the walls of a compact domain in the plane bounded by finitely many straight lines making angles which are rational in units of π. We will consider here triangles, henceforth called rational triangles. The properties of the classical motion in such triangles have been the subject of numerous works which can partly be traced from reference [2].

Our interest is primarily in the corresponding quantum problem, namely to find the spectrum of (minus) the Laplace operator with Dirichlet boundary conditions. In collaboration with J.M. Luck and P. Moussa [3] one of the authors has written sum rules for

Key-Words: Rational billiards, algebraic curves.
1980 MSC: 14 H, 70 H, 81 F.

(*) Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique.

powers of the inverse eigenvalues. On the other hand M. Gaudin [4] has developed a method which would also give access to the eigenfunctions.

The present note is only devoted to a minor aspect of the problem, namely we want to show that such a rational triangle is «canonically» associated to an algebraic curve up to birational equivalence (the natural model possesses singularities). This curve has in general a large group of analytic automorphisms. The corresponding Riemann surface is paved by a finite number of equivalent triangles. Conversely this curve of genus g allows us to define a correspondence between g rational triangles (we call them associated) using a privileged basis in the space of holomorphic differentials. This suggests the (wild) speculation that this correspondence could extend to algebraic relations between appropriate spectral quantities.

As the following discussion will demonstrate, we have no claims at being experts in algebraic geometry. We are therefore not sure that our terminology is quite correct nor that our observations have not already been reported elswhere. In preparing this note we have benefited from many discussions with J. Lascoux and J.M. Luck. We thank them warmly here.

2. When a particle bounces elastically at the boundary of the biliard, its perpendicular velocity changes sign (we exclude singular trajectories which hit the corners). The triangular billiard is characterized by its angles θ_i, (i taking values $0, 1, \infty$) which are rational in units of π, so that for pairs of coprime positive integers p_i, q_i, with $q_i > 0$, we have

$$\theta_i = \pi \frac{p_i}{q_i}, \qquad (p_i, q_i) = 1, \qquad \frac{p_0}{q_0} + \frac{p_1}{q_1} + \frac{p_\infty}{q_\infty} = 1 \tag{1}$$

Let Q denote the lowest common multiple of q_0, q_1, q_∞ (in fact of any pair). The velocity of the particle can take at most $2Q$ values (of equal length). Let the dihedral group D_{2Q} generated by the reflections in two lines at angle π/Q act on the hodograph (the circle S_1 describing the directions of the velocity). The phase space is then foliated by closed surfaces made of $2Q$ copies of the original triangle glued along their common sides according to the action of this group. This compact (orientable) surface is therefore paved by $n_2 = 2Q$ triangles (which could alternatively be painted in two colors) having in common $n_1 = 3Q$ sides. Moreover according to a reasoning due to Riemann and Hurwitz each vertex (i) of the original triangle is the projection of Q/q_i distinct vertices on this surface. The Euler characteristic χ and the genus g are therefore related to these numbers through

$$\chi = 2 - 2g = n_0 - n_1 + n_2 = \left(\sum_i \frac{Q}{q_i}\right) - Q = \sum_i \left[\frac{Q}{q_i}(1 - p_i)\right] \tag{2}$$

By extension we also call g the genus of the triangle. The condition of integrability $\chi = 0$ or $g = 1$, corresponding to a torus, is therefore that the numerator p_i of the irreducible fractions be equal to one. There exist only three solutions in one to one correspondence with the simple Lie algebras (over $\mathbb{C}$) of rank 2, denoted $A_2, B_2 \sim C_2$ and G_2 [5]. Up to scale, the triangle is then a fundamental domain for the inhomogeneous Weyl group or Coxeter group associated to the characters of the corresponding compact Lie group and the spectrum is given in terms of the quadratic Casimir invariant [6]. In some loose sense the study of rational triangles appears as a hyperbolic generalisation of this integrable case.

3. It is convenient to replace the three pairs of integers p_i, q_i by three integers P_i with no common factor such that

(1) $$P_0 + P_1 + P_\infty = Q \quad p_i/q_i = P_i/Q \quad \ell_i = (P_i, Q) = P_i/p_i = Q/q_i$$

We label then the triangle by $\{P_0, P_1, P_\infty\}$. Up to ordering of the vertices we show in the following table the first few triangles classified according to genus (up to $g = 5$)

	P_0	P_1	P_∞	
$g=1$	1	1	1	A_2
	2	1	1	B_2
	3	2	1	G_2
$g=2$	2	2	1	
	3	1	1	
	4	1	1	
	4	3	1	
	5	3	2	
	5	4	1	
$g=3$	3	2	2	
	3	3	1	
	4	2	1	
	5	1	1	
	3	3	2	
	5	2	1	
	6	1	1	
	4	3	2	
	5	3	1	
	6	2	1	
	5	4	3	
	6	5	1	
	8	3	1	
	7	4	3	
	7	5	2	
	7	6	1	

	P_0	P_1	P_∞
$g=4$	4	4	4
	5	2	2
	7	1	1
	4	3	3
	6	3	1
	7	2	1
	8	1	1
	7	3	2
	7	4	1
	9	2	1
	6	5	4
	7	5	3
	9	5	1
	10	3	2
	8	5	3
	8	7	1
	9	5	4
	9	7	2
	9	8	1

	P_0	P_1	P_∞
$g=5$	4	4	3
	5	3	2
	5	4	2
	5	5	1
	6	3	2
	6	4	1
	7	2	2
	7	3	1
	8	2	1
	9	1	1
	5	5	2
	10	1	1
	8	5	2
	10	4	1
	10	7	3
	10	9	1
	11	6	5
	11	7	4
	11	8	3
	11	9	2
	11	10	1

Observe that

LEMMA. The integer $Q = P_0 + P_1 + P_\infty$ and the genus g satisfy the inequalities

$$2g + 1 \leq Q \leq 2(2g + 1) \tag{2}$$

The first inequality follows from the definition of g written in the form

$$Q = 2g + 1 + \sum_i (\ell_i - 1) \tag{3}$$

with $\ell_i = (Q, P_i)$. It is only satured when all $\ell_i = 1$, so that Q is odd and one can take as an example the triangle $\{1, g, g\}$.

To prove the second inequality, we observe that the three integers $\ell_i = (Q, P_i)$ are without common factors since the three P_i 's are. For each prime p there exists at least one ℓ_i prime to p and therefore a pair $Q, q_i = Q/\ell_i$ which admits the same power of p as divisor. This same power of p divides the product $q_j q_k$ since it divides the lowest commun multiple Q of the pair q_j, q_k. Therefore Q^2 divides $q_0 q_1 q_\infty$. As a consequence

$$Q \geq \frac{Q}{q_0} \cdot \frac{Q}{q_1} \cdot \frac{Q}{q_\infty} = \ell_0 \ell_1 \ell_\infty .$$

Let us order the three integers ℓ_i, for instance $l_0 \geq \ell_1 \geq \ell_\infty$. Set $q_0 = Q/p_0 = m \geq 2$. The above inequality reads $\ell_1 \ell_\infty \leq m$ therefore $\ell_1 + \ell_\infty \leq m + 1$. Consequently

$$\ell_0 + \ell_1 + \ell_\infty \leq \frac{Q}{m} + m + 1$$

for m a divisor of Q larger or equal to 2. If $m = Q$ we have $\ell_0 = \ell_1 = \ell_\infty$ and we recover the lower bound. Let us therefore assume $2 \leq m \leq Q/2$. In this interval the function $\frac{Q}{m} + m + 1$ assumes its maximum $Q/2 + 3$ at both ends, and $\ell_0 + \ell_1 + \ell_\infty \leq \frac{Q}{2} + 3$ or equivalently $\sum(\ell_i - 1) \leq Q/2$. Inserting this in equation (3) yields the second inequality of the lemma. It can only be satured for Q even, in which case an example is the triangle $\{1, 2g, 2g + 1\}$.

It follows that the number N_g of triangles (up to permutations of vertices) of a given genus is finite. It would be easy to obtain upper bounds on N_g. We were unable to find a simple closed expression for this quantity.

4. In reference [3] we used the observation that the Schwarz conformal one-to-one map from any triangle to the upper half plane enables one to obtain the Green function for the Dirichlet problem in the triangle. Here we study this map in more detail for rational triangles. It applies the boundary of the triangle on the real axis and the three vertices on the points $0, 1, \infty$, for instance (hence the labeling). Up to translation, rotation, or rescaling, we write

$$t = \int_0^x \frac{\mathrm{d}\, x'}{x'^{1-P_0/Q}(1-x')^{1-P_1/Q}} \tag{1}$$

where t is a complex coordinate in the plane of the triangle, and x is in the half plane $\operatorname{Im} x > 0$. The contour of integration joins 0 to x in this half plane, and for definiteness the integrand tends to a positive real value as x tends to a point in the segment $(0, 1)$. The scale is therefore fixed by the length of the side

$$|t_1 - t_0| = \frac{\Gamma\left(\frac{P_0}{Q}\right)\Gamma\left(\frac{P_1}{Q}\right)}{\Gamma\left(\frac{P_0+P_1}{Q}\right)}. \tag{2}$$

The area of the triangle is

$$A = \frac{\pi}{2}\frac{\Gamma\left(\frac{P_0}{Q}\right)\Gamma\left(\frac{P_1}{Q}\right)\Gamma\left(\frac{P_\infty}{Q}\right)}{\Gamma\left(\frac{P_1+P_\infty}{Q}\right)\Gamma\left(\frac{P_\infty+P_0}{Q}\right)\Gamma\left(\frac{P_0+P_1}{Q}\right)}. \tag{3}$$

According to formula (1) the derivative

$$y = \frac{\mathrm{d}\, x}{\mathrm{d}\, t} \tag{4}$$

is related to x by the algebraic equation

$$y^Q = x^{P_1+P_\infty}(1-x)^{P_\infty+P_0} \tag{5}$$

canonically associated to the triangle up to permutations of its vertices. The latter are generated by the transformations

$$\begin{array}{ll} 0 \leftrightarrow 1 & (x, y) \to (1-x, y) \\ 0 \leftrightarrow \infty & (x, y) \to \left(x^{-1},\ e^{i\pi\left(1-\frac{P_1}{Q}\right)} y x^{-2}\right) \end{array} \tag{6}$$

the square of which are automorphisms of the curve. It is readily checked that the curve (5) has a genus given by the same formula (2-2) and that the corresponding connected Riemann surface can be considered as paved by $2Q$ triangles, each one the pre-image of the upper or lower half x-plane (with the points $0, 1, \infty$, marked).

Conversely, given an irreducible curve such as $y^Q = x^{S_0}(1-x)^{S_1}$ with Q, S_0, S_1 positive integers without common factor, provided $P_0 = Q - S_0, P_1 = Q - S_1$ and $P_\infty = S_0 + S_1 - Q$ are all positive integers, it defines a corresponding triangle and y and x allow one to obtain $\mathrm{d}t$ as $\mathrm{d}x/y$.

Each point $x = 0, 1, \infty$, is a singular point of the curve in the vicinity of which one can introduce a regularizing parameter u_0, u_1, u_∞ such that for each of the ℓ_i families of $q_i = Q/\ell_i$ sheets, one has (with $p_i = P_i/\ell_i$)

$$(7)\qquad \begin{array}{lll} (0) & y \sim u_0^{q_0 - p_0} & x \sim u_0^{q_0} \\ (1) & y \sim u_1^{q_1 - p_1} & (1-x) \sim u_1^{q_1} \\ (2) & y \sim e^{i\pi\left(1-\frac{p_1}{q_1}\right)} u_\infty^{-q_\infty - p_\infty} & x \sim u_\infty^{-q_\infty} \end{array}$$

It follows that x and y are meromorphic functions on the Riemann surface. The differential

$$(8)\qquad \omega_{(1)} = \mathrm{d}t = \frac{\mathrm{d}x}{y}$$

is obviously regular at each pair (x, y) satisfying (5) such that x is different from $0, 1, \infty$. At these points one easily verifies using (7) that it continues to be regular since

$$(9)\qquad \begin{array}{ll} (0) & \omega_{(1)} \sim q_0 u_0^{p_0 - 1}\, \mathrm{d}u_0 \\ (1) & \omega_{(1)} \sim -q_1 u_1^{p_1 - 1}\, \mathrm{d}u_1 \\ (\infty) & \omega_{(\infty)} \sim -q_\infty e^{-i\pi\left(1-\frac{p_1}{q_1}\right)} u_\infty^{p_\infty - 1}\, \mathrm{d}u_\infty \end{array}$$

The differential $\omega_{(1)}$ can only vanish at the pre-images of $x = 0, 1, \infty$, and in these $\ell_0, \ell_1, \ell_\infty$, points it vanishes with multiplicities $p_0 - 1, p_1 - 1$ and $p_\infty - 1$ respectively, its total numper of zeroes is therefore $\sum \ell_i(p_i - 1) = 2(g-1)$, a particular case of a general theorem.

5. When the genus is unity, the holomorphic differential $\omega_{(1)}$ is unique up to a factor and the corresponding abelian integral, the variable t, plays the role of uniformizing parameter. The complex t-plane is the universal covering space of the surface. The correspondence between integrable triangles and elliptic curves reads

$$(10)\qquad \begin{array}{llll} \text{(a)}\ \{1,1,1\} & y^3 = [x(1-x)]^2 & \text{modulus} & \tau = e^{2i\pi/3} \\ \text{(b)}\ \{1,1,2\} & y^4 = [x(1-x)]^3 & & \tau = e^{2i\pi/4} \\ \text{(a)}\ \{1,2,3\} & y^6 = x^5(1-x)^4 & & \tau = e^{2i\pi/6} \sim e^{2i\pi/3} \end{array}$$

One could easily express x and y as doubly periodic meromorphic functions of t. Of course these curves can be desingularized. For instance, under

$$X = \frac{x(1-x)}{y} \qquad Y = 1 - 2x \qquad x = \frac{1-Y}{2} \qquad y = \frac{1-Y^2}{4X} \tag{11}$$

the first takes the canonical form

$$\text{(a)} \qquad Y^2 + 4X^3 = 1 \tag{12}$$

the second

$$\text{(b)} \qquad Y^2 + 4X^4 = 1 \tag{13}$$

For the third curve we write

$$X = \frac{y^2}{x^2(1-x)} \qquad Y = \frac{x^2(1-x)^2}{y^3} \qquad x = \frac{1}{Y^2} \qquad y = \frac{Y^2-1}{XY^3} \tag{14}$$

such that

$$\text{(c)} \qquad Y^2 = X^3 + 1 \tag{15}$$

which up to scale $(X \to -2^{2/3}X)$ agrees with the first.

When the genus g is higher than one, a Riemann surface admits g linearly independent holomorphic differentials (Abelian differentials of first kind). For the curve (4.5) $\omega_{(1)}$ is one of them. It is interesting to see whether one can choose a basis of such differentials in such a way that their integrals map the upper half plane $\mathrm{Im}\, x > 0$ on a rational triangle. We shall call such triangles associates.

6. Let therefore P_0, P_1, P_∞, be positive integers with no common factor. We set $Q = P_0 + P_1 + P_\infty$ and call $\mathcal{T}$ the corresponding triangle with corresponding irreducible curve (4.5). It will sometimes be convenient to label a triangle by three integers having a common factor, in which case the integers P_i will be understood after division by this factor. To define the associates, we consider for an integer v the three integers vP_0, vP_1, vP_∞, and take their representatives modulo Q. More precisely we write

$$0 \le P_i^{(v)} \le Q - 1 \qquad P_i^{(v)} \equiv vP_i \bmod Q \tag{1}$$

Among the $Q-1$ values $1 \le v \le Q-1$, we call admissible those for which

$$P_i^{(v)} > 0 \qquad P_i^{(v)} + P_1^{(v)} + P_\infty^{(v)} = Q \tag{2}$$

which means of course that $P_0^{(v)}, P_1^{(v)}$ and $P_\infty^{(v)}$ define a triangle $\mathcal{T}^{(v)}$ with angles $\pi P_i^{(v)}/Q$. Thus $\mathcal{T}^{(1)} = \mathcal{T}$ and for v admissible $\mathcal{T}^{(v)}$ is an associate triangle. Note that it is not garanteed that the $P_i^{(v)}$ have no common factor.

PROPOSITION 1. *The number of admissible values of v, hence the number of associates of a triangle $\mathcal{T}$ of genus g, is precisely g.*

The steps in the proof are as follows.

(i) Two distinct values of v in the interval $[1, Q-1]$ cannot lead to the same triangle in the sense that $v_1 \neq v_2$ yield two distinct ordered triplets $\{P_0^{(v_i)}, P_1^{(v_i)}, P_\infty^{(v_i)}\}$. Indeed if $v_2 > v_1$ for instance, then $v_2 - v_1 \leq Q - 2$. If the triplets were identical we should have $(v_2 - v_1) P_i = k_i Q$, thus $P_i/Q = k_i/(v_2 - v_1)$ in contradiction with the fact that Q is the lowest common denominator of the fractions P_i/Q and $v_2 - v_1$ is smaller than Q. Two such distinct ordered triplets which both add to Q cannot obviously be (distinct) multiples of a same reduced ordered triplet without common factor.

(ii) Both of the integers v and $Q - v$ in the range $[1, Q-1]$ cannot be simultaneously admissible. For each value $i, P_i^{(v)}$ and $P_i^{(Q-v)}$ are either both zero or both non zero. If all $P_i^{(v)}$ are positive (and smaller than Q) so are all $P_i^{(Q-v)} = Q - P_i^{(v)}$. Thus

$$(P_0^{(Q-v)} + P_1^{(Q-v)} + P_\infty^{(Q-v)}) + (P_0^{(v)} + P_1^{(v)} + P_\infty^{(v)}) = 3Q \tag{3}$$

and one of the sums is Q the other $2Q$. Thus at most one of the values v and $Q - v$ is admissible.

(iii) If one the $P_i^{(v)}$ vanishes, we have for this value of i, $vP_i = k_i Q$ so that dividing both sides by $\ell_i = (P_i, Q)$ we conclude that $q_i = Q/\ell_i$ divides v. Conversely if v is a multiple of $q_i, P_i^{(v)}$ vanishes.

(iv) It follows that the admissible values of v are obtained from the following algorithm. Erase from the set of integers $1, \ldots, Q-1$, the multiples of q_0, q_1, q_∞. If v is among the remaining integers so is $Q - v$, since v and $Q - v$ have opposite residues modulo q_i. For the remaining v's the $P_i^{(v)}$ are all positive, and $\sum_i P_i^{(v)}$ is Q or $2Q$ while correspondingly $\sum_i P_i^{(Q-v)}$ is $2Q$ or Q according to (3). It therefore follows that either v or $Q - v$ is admissible. The number of multiples of q_i in the range $(1, Q-1)$ is $\ell_i - 1$. In this range, since Q is the least common multiple of the q_i's, no multiple of q_i is a multiple of a $q_j, j \neq i$. The number of admissible v's is therefore

$$\frac{1}{2}\{Q - 1 - \sum_i (\ell_i - 1)\} = g \tag{4}$$

where the last equality follows from the definition (2-2), proving the proposition. ∎

7. Let us return to the irreducible algebraic curve C given by $y^Q = x^{P_1+P_\infty}(1-x)^{P_\infty+P_0}$ and for v admissible let $y^Q_{(v)}$ be defined through

$$y^Q_{(v)} = x^{P_1^{(v)}+P_\infty^{(v)}}(1-x)^{P_\infty^{(v)}+P_0^{(v)}}. \tag{5}$$

Note that if the $P_i^{(v)}$'s have a common factor this curve $C_{(v)}$ will not be irreducible. Let us also define

$$\omega_{(v)} = \frac{\mathrm{d}x}{y_{(v)}} \tag{6}$$

where of course $y_{(1)} = y$ and $\omega_{(1)}$ is the differential defined above.

PROPOSITION 2. *For v admissible*

(i) $y_{(v)}$ *is a meromorphic function on the Riemann surface attached to the curve C. More precisely a rational function $y_{(v)}$ of y and x can be defined satisfying (5).*

(ii) *The differential $\omega_{(v)}$ is holomorphic on the Riemann surface and maps the upper half x-plane on the corresponding associated triangle.*

(iii) *When v ranges over admissible values, the g holomorphic differentials $\omega_{(v)}$ are linearly independent.*

Since there exist only g holomorphic linearly independent differentials, the proposition yields an explicit construction. The simple proof is as follows.

(i) Write $P_i^{(v)} = vP_i - r_iQ$ with r_i a non negative integer, $P_i^{(v)}$ in the range $[1, Q-2]$ and $\sum P_i^{(v)} = Q$. With these notations it is readily verified that

$$y_{(v)} = y^v x^{r_0+1-v}(1-x)^{r_1+1-v} \tag{6}$$

satisfies equation (5).

(ii) An immediate calculation shows that in the vicinity of $x = 0, 1, \infty$, using the parameters introduced in (4.7) one has up to numerical coefficients

$$(7) \quad \begin{aligned} &(0) \quad y_{(v)} \sim u_0^{\frac{Q-P_0^{(v)}}{\ell_0}} \qquad \omega_{(v)} \sim u_0^{\frac{P_0^{(v)}}{\ell_0}-1}\,\mathrm{d}u_0 \\ &(1) \quad y_{(v)} \sim u_1^{\frac{Q-P_1^{(v)}}{\ell_1}} \qquad \omega_{(v)} \sim u_1^{\frac{P_1^{(v)}}{\ell_1}-1}\,\mathrm{d}u_1 \\ &(\infty) \quad y_{(v)} \sim u_\infty^{-\frac{Q+P_\infty^{(v)}}{\ell_0}} \qquad \omega_{(v)} \sim u_\infty^{\frac{P_\infty^{(v)}}{\ell_\infty}-1}\,\mathrm{d}u_\infty \end{aligned}$$

Note that one can now only say that ℓ_i divides $(Q, P_i^{(v)})$. The above formulas show that at any preimage of the singular projections $x = 0, 1, \infty$, the differential $\omega_{(v)}$ is regular. Of course

$$\sum_i \ell_i \left(\frac{P_i^{(v)}}{\ell_i} - 1 \right) = 2(g-1).$$

(iii) It remains to prove linear independence. Let $\sum_v$ stands for the sum over admissible v 's, and assume a linear relation of the form

$$\sum_v a_v \omega_{(v)} = 0$$

with constant coefficients not all of them vanishing. It would follow that

$$\sum_v \frac{a_v}{y_v} = 0$$

and from equation (6) (where r_0 and r_1 depend on v) and after multiplication by y^Q

$$\sum_v a_v y^{Q-v} x^{v-1-r_0^{(v)}} (1-x)^{v-1-r_1^{(v)}} = 0.$$

Multiplying by a common factor, we obtain a relation of the form

$$\sum_v y^{Q-v} p_v(x) = 0$$

with $p_v(x)$ a polynominal in x, and the positive exponent of y is smaller than Q. This would contradict the irreducibility of the curve $y^Q - x^{P_1+P_\infty}(1-x)^{P_\infty+P_0} = 0$, and proves the proposition since it is readily seen that the integral of $\omega_{(v)}$ maps the upper half x-plane on the corresponding triangle $\mathcal{T}^{(v)}$. ∎

8. It is nice to examine an example. Consider the triangle (113) of genus 2 with angles $\pi/5, \pi/5, 3\pi/5$. If one reflects this triangle along the sides intersecting at the point marked zero, one obtains the star shown on figure 1-a. With proper identification of the free sides, this gives rise to the triangulated double torus shown on figure 1-b, as a topological model. In this case $q_0 = q_1 = q_\infty = Q = 5$ and the corresponding curve C is

$$(1) \qquad y_{(1)}^5 = [x(1-x)]^4$$

The admissible values of v are 1 and 2, with $\mathcal{T}^{(1)} \equiv \mathcal{T}$ and $\mathcal{T}^{(2)}$ is the triangle $\{221\}$ of angles $2\pi/5, 2\pi/5, \pi/5$ (the two triangles are sometimes called Robinson

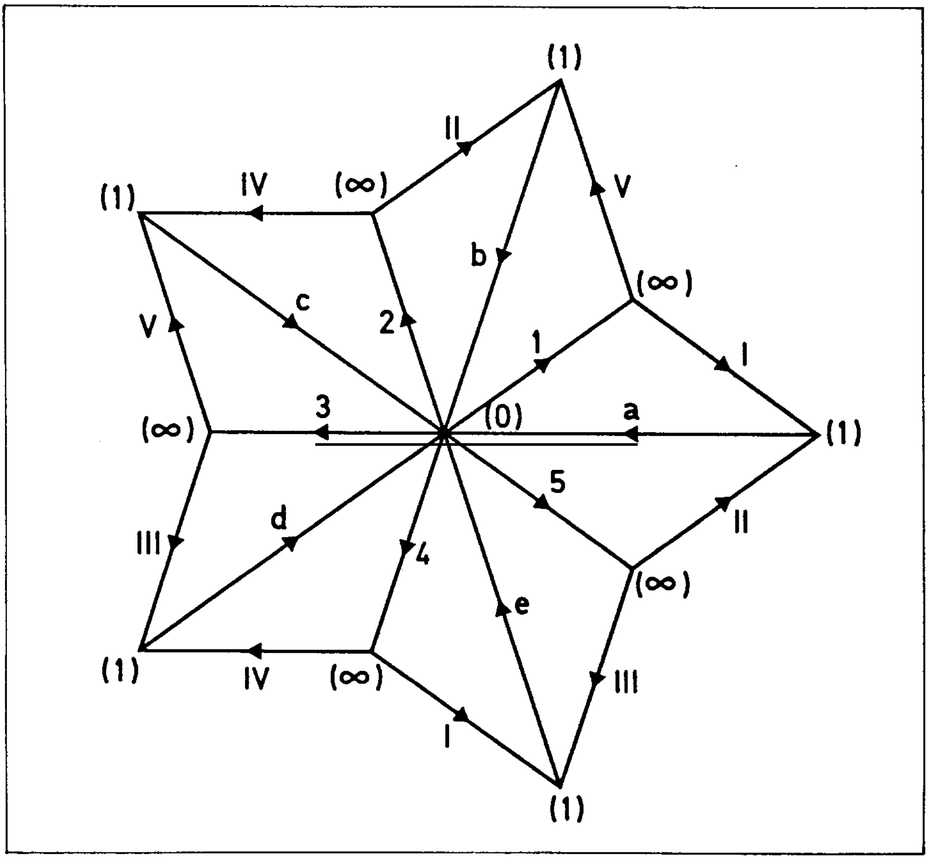

Figure 1-a

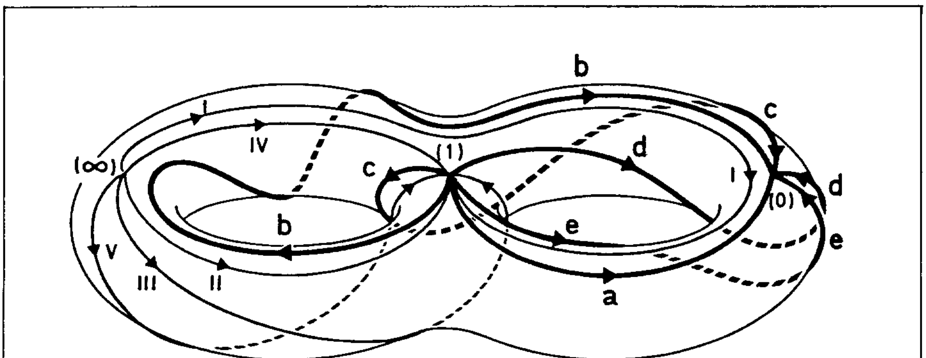

Figure 1-b

triangles). By completing each of these triangles by a reflected one in one of its sides, one obtains the motives which appear in Penrose's tiling of the plane. The second triangle of the same genus 2 is attached to the curve

$$y_{(2)}^5 = [x(1-x)]^3 \tag{2}$$

birationally equivalent to (1). Both are birationally equivalent to the non singular curve

$$Y^2 + X^5 = 1 \tag{3}$$

and possess an automorphism group $Z/2Z \times Z/5Z$.

To exhibit this equivalence without using irrationalities, we rescale $y_{(1)}$ and $y_{(2)}$ (by factors $4^{-4/5}$ and $4^{-3/5}$ respectively) so that one obtains the following table

$$Y^2 + X^5 = 1$$

$$X = \frac{y_{(1)}^4}{[4x_{(1)}(1-x_{(1)})]^3} \qquad X = \frac{[4x_{(2)}(1-x_{(2)})]^2}{y_{(2)}^3}$$

$$Y = 1 - 2x_{(1)} \qquad Y = 1 - 2x_{(2)}$$

$$x_{(1)} = \frac{1-Y}{2} \quad x_{(2)} = \frac{1-Y}{2}$$

$$y_{(1)} = X^4 \quad y_{(2)} = X^3$$

$$x_{(1)} = x_{(2)} \qquad y_{(1)} = \left[\frac{4x_{(2)}(1-x_{(2)})}{y_{(2)}}\right]^2$$

$$y_{(1)}^{(5)} = [4x_{(1)}(1-x_{(1)})]^4 \rightleftarrows y_{(2)}^5 = [4x_{(2)}(1-x_{(2)})]^3$$

$$x_{(2)} = x_{(1)} \qquad y_{(2)} = \frac{y_{(1)}^2}{4x_{(1)}(1-x_{(1)})}$$

One can consider that the Riemann surface of the curve $Y^2 + X^5 = 1$ is endowed with a triangulation (by 10 triangles) such that, when embeded in its Jacobian variety parametized by the integrals of $\omega_{(1)}$ and $\omega_{(2)}$, each triangle is projected on a Robinson triangle on each coordinate complex plane. If t_1 and t_2 are the respective variables, it would be interesting to describe the codes which link these two planes to obtain a Penrose tiling. By following the classical geodesic motion in one of the triangles lifted to the

Riemann surface one can also obtain information about geodesics (like their homology class ...).

9. In the general case, let us investigate certain peculiarities of associated triangles. A first interesting possibility is when $\mathcal{T}$ and $\mathcal{T}'$ only differ by a transposition of two vertices. Assume $\mathcal{T} \equiv \mathcal{T}^{(1)}, \mathcal{T}' \equiv \mathcal{T}^{(v)}$. A necessary condition is then that for $i = 0, 1, \infty$, and v an admissible value larger than one

$$(1) \qquad (v^2 - 1)P_i \equiv P_i \qquad \bmod Q$$

equivalently $(v^2 - 1)p_i = k_i q_i$, which means that q_i divides $(v^2 - 1)$. This being true for any i, Q the lowest common multiple of the q_i 's divides $v^2 - 1$

$$(2) \qquad (v^2 - 1) \equiv 0 \qquad \bmod Q.$$

One of the vertices is common to $\mathcal{T}$ and $\mathcal{T}'$, let it be P_0 for instance. Then $vP_0 \equiv P_0 \bmod Q$, $vP_1 \equiv P_\infty \bmod Q$ and $vP_\infty \equiv P_1 \bmod Q$. Any common divisor of v and Q would then divide P_0, P_1, P_∞, which is only possible for unity, thus

$$(3) \qquad (Q, v) = 1.$$

Furthermore since $(v - 1)P_0 = kQ$ this implies that $\ell_0 = (Q, P_0) > 1$, since Q prime to P_0 would require that Q divides $v - 1 < Q$. Thus

$$(4) \qquad \ell_0 = (Q, P_0) > 1.$$

If these conditions are satisfied, and $\mathcal{T}$ and $\mathcal{T}^{(v)}$ differ by a transposition of vertices, the $(g - 2)$ remaining triangles either are in pairs differing by a transposition of the corresponding two vertices, or else are isoscele or equilateral triangles. For instance in genus 5 the triangle $\{2, 5, 8\}$ is associated to $\{8, 5, 2\}$ for $v = 4$ $(v^2 - 1 = 15 = Q)$ and to the pair $\{4, 10, 1\}$ and $\{1, 10, 4\}$ for $v = 2$ and 8, but for $v = 5$ $((Q, v) = 5)$ one finds the equilateral triangle $\{1, 1, 1\}$ «covered» ten times (see below).

A simpler example is given by the triangle $\{4, 3, 1\}$ of genus 2 and $Q = 8$. The associated triangle obtained for $v = 3$ is $\{4, 1, 3\}$. We have P_0 fixed, $\ell_0 = 4$, and $v^2 - 1 = 8$. The two equivalent curves are

$$(5) \qquad \begin{aligned} &y^8 = x^4(1 - x)^5 \\ &y_{(3)}^8 = x^4(1 - x)^7 \qquad y_{(3)} = \frac{y^3}{x(1 - x)}. \end{aligned}$$

A desingularized canonical form is the curve

$$(6) \qquad Y^2 = X(1 - X^4)$$

given by the relations

$$(7)\qquad \begin{aligned} X &= \frac{y^2}{x(1-x)} & x &= 1 - X^4 \\ Y &= \frac{x^2(1-x)^2}{y^3} & y &= X^2 Y, \quad Y^{(3)} = X^3 Y. \end{aligned}$$

This is the hyperelliptic curve of genus two with the largest possible automorphism group of order 48, a double covering of the octaedral group. The octaedron has its vertices on the Riemann sphere of the variable X at the six points $0, \infty, \pm 1, \pm i$, which project in the x variable at $1, \infty$ and 0 (four times). The differential $\omega_{(1)}$ and $\omega_{(3)}$ are respectively $\frac{-4X\,dX}{Y}$ and $\frac{-4\,dX}{Y}$. The triangulation in 16 triangles appears naturally as the double covering of the one obtained on the X-Riemann sphere by projecting the faces of the octaedron.

An other case of interest corresponds to the possibility that associated triangles have circularly permuted vertices. Let these triangles be $\mathcal{T}^{(1)}$ and $\mathcal{T}^{(v)}$ as before. This requires

$$(8)\qquad 1 + v + v^2 \equiv 0 \qquad \text{mod } q_i$$

for each value of i. From $vP_0 = P_1 + k_1 Q \quad v^2 P_0 = P_\infty + k_2 Q$ it follows that ℓ_0 divides P_1 and P_∞ which is impossible unless $\ell_0 = 1$ and similarly $\ell_1 = \ell_\infty = 1$. Thus all q_i are equal to Q,

$$(9)\qquad 1 + v + v^2 \equiv 0 \qquad \text{mod } Q$$

and the fractions with denominator Q are irreducible. Note that Q must be odd. If Q is prime, the triangles are then associated in triplets and Q is of the form $Q \equiv 1 \bmod 6$. For instance the triangle $\{1,2,4\}$ of genus 3 has $Q = 7$ and for the admissible values $v = 1,2,4$ (with $1 + 2 + 2^2 \equiv 0 \bmod 7$) is associated to $\{1,2,4\}$, $\{2,4,1\}$ and $\{4,1,2\}$. Similarly for $Q = 13$ the associates of the genus 6 triangle $\{1,3,9\}$ are

$$(10)\qquad \begin{array}{rlrl} v=1 & \{1,3,9\} & v=2 & \{2,6,5\} \\ 3 & \{3,9,1\} & 5 & \{5,2,6\} \\ 9 & \{9,1,3\} & 6 & \{6,5,2\} \end{array}$$

If however Q is not a prime, $Q - 1$ is not necessarily a multiple of 6 as shown by the example of the associates of the triangle $\{1,4,16\}$ of genus 10 with $Q = 21$. The admissible values of v are classified as follows

$$(11)\qquad \begin{array}{rlrl} v=1 & \{1,4,16\} & v=2 & \{2,8,11\} \\ 4 & \{4,16,1\} & 8 & \{8,11,2\} \\ 16 & \{16,1,4\} & 11 & \{11,2,8\} \\ & & & \\ v=3 & \{3,12,6\} & v=7 & \{7,7,7\} \\ 6 & \{6,3,12\} & & \\ 12 & \{12,6,3\} & & \end{array}$$

For $v = 3, 6, 12$, the integers $P_i^{(v)}$ have a common factor 3 and up to cyclic permutation the corresponding triangle is $\{1, 4, 2\}$ of genus 3 (and not 10) discussed above, while for $v = 7$ one obtains in irreducible terms the equilateral triangle $\{1, 1, 1\}$ of genus one.

This gives an example of a correspondence between algebraic curves of genus larger or equal to one. In the preceding example starting from the irreducible curve

$$(12) \qquad g = 10 \qquad y^{21} = x^{20}(1 - x)^{17}$$

we define the correspondence deduced from equation (7-6) $y_{(7)} = Y$ $x_{(7)} = X$ such that

$$(13) \qquad Y = \frac{y^7}{x^6(1 - x)^5} \qquad X = x$$

and we obtain the elliptic curve

$$(14) \qquad g = 1 \qquad Y^3 = [X(1 - X)]^2$$

of modular ratio $\tau = e^{2i\pi/3}$. Thus instead of considering the Riemann surface of the curve (12) as ramified above the Riemann sphere of the $X = x$ variable we can also think of it as a ramified 7-uple covering of the torus given by (14).

This same curve can also be interpreted as a triple covering of the curve genus 3

$$(15) \qquad g = 3 \quad Y'^7 = X'^6(1 - X')^3, \qquad X' = x \quad Y' = \frac{y^3}{[x(1 - x)]^2}.$$

This phenomenon occurs for Q non prime and for v admissible when the $P_i^{(v)}$'s have a common factor which lowers the genus.

Let us also give the example of the triangle $\{4, 4, 7\}$ of genus 7, the associates of which in reduced terms are $\{1, 1, 13\}$, $\{1, 1, 1\}$, $\{2, 2, 11\}$, $\{2, 2, 1\}$, $\{1, 1, 3\}$, and $\{7, 7, 1\}$. On figure 2 we show how the corresponding surface can be visualized as a 15 studded star (associated to $\{1, 1, 13\}$ with identification of sides) or five copies of a hexagon (associated to the equilateral triangle $\{1, 1, 1\}$) or three copies of a 5-studded star (associated to the Robinson triangle $\{1, 1, 3\}$). Again we have analytic maps between Riemann surfaces of different genus. J. Lascoux has drawn our attention to a theorem of de Franchis showing that for a pair of compact Riemann surfaces of genus higher that one, such mapings are finite in number (and in general this number is zero). The «triangular» curves discussed in this paper offer a rich zoo of examples to study such maps and their relations to automorphism groups of curves.

10. The curves associated to rational triangles possess striking algebraic properties. Their uniformization involves a Fuchsian group related to the Coxeter group generated by the reflections in the sides of the triangles. Coxeter groups of associated triangles are either isomorphic or one is a factor of the former. The incorporation of these properties would perhaps allow to get new insight into the theory of pseudo-integrable systems.

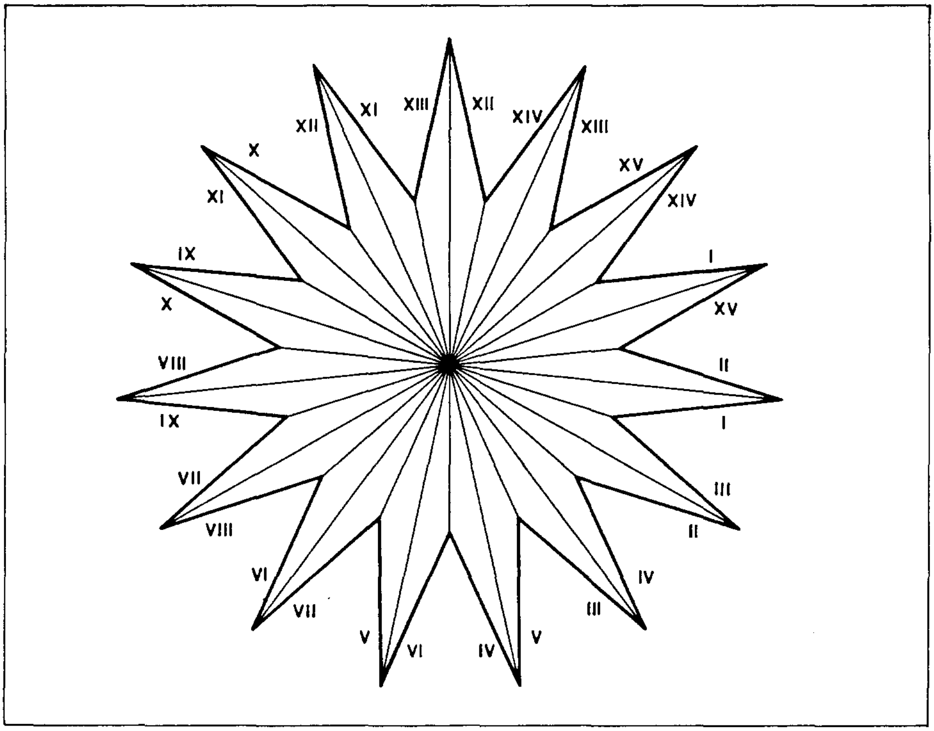

Figure 2-a

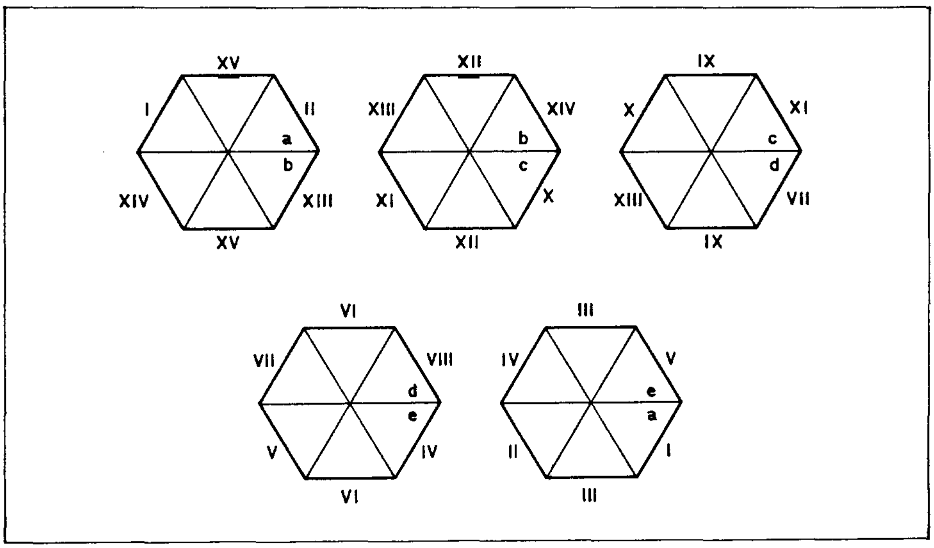

Figure 2-b

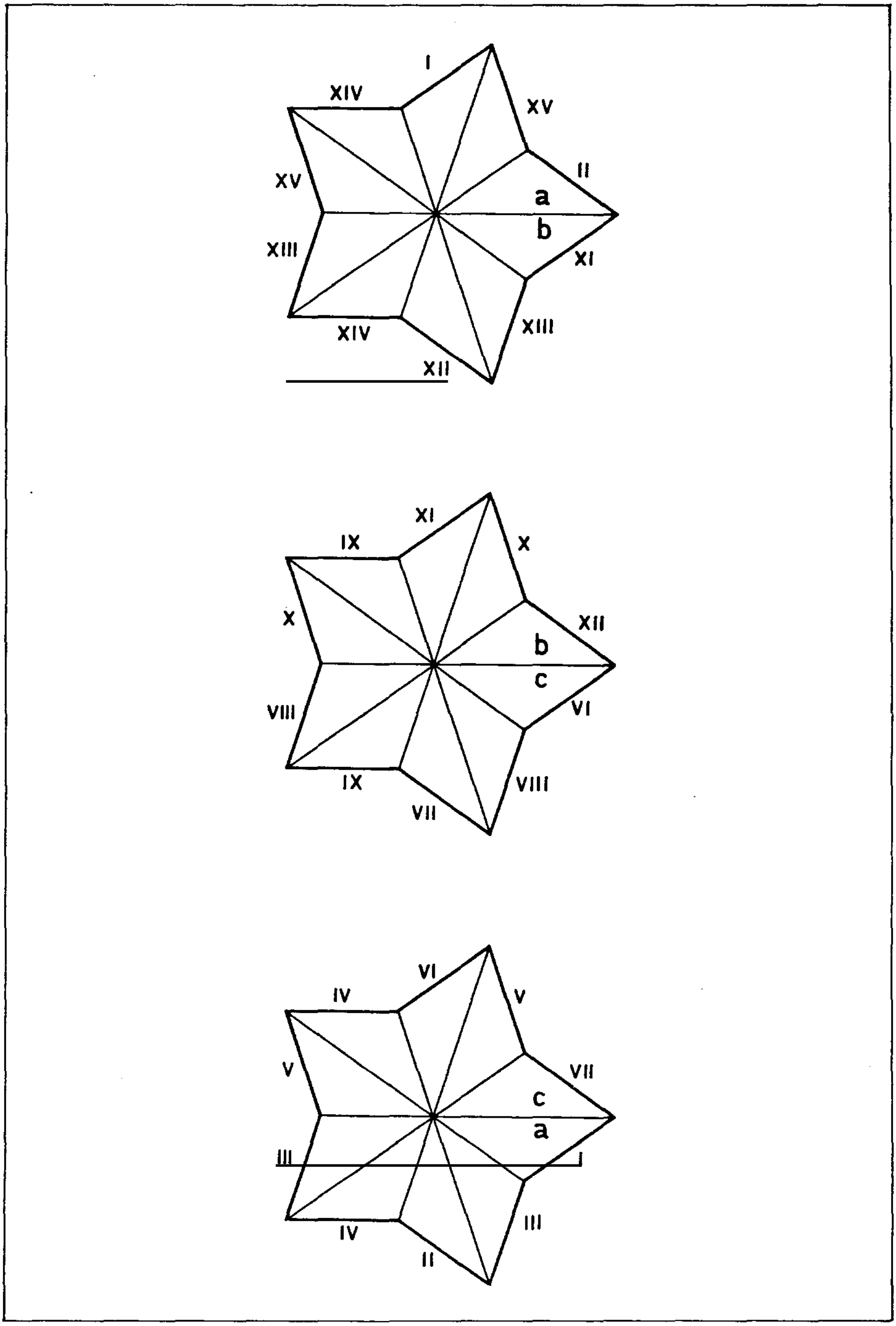
I
XIV
XV
XV
II
a
b
XIII
XI
XIV
XIII
XII
XI
IX
X
X
XII
b
c
VIII
VI
IX
VIII
VII
VI
IV
V
V
VII
c
a
III
I
IV
III
II

Figure 2-c

ACKNOWLEDGEMENTS

One of the authors (C.I.) would like to thank warmly P. Cohen and J. Wolfart with whom he had the pleasure to discuss this work after its completion, and who made him aware of its connection with complex multiplication types of the Jacobian varieties of Fermat curves deduced from the work of Koblitz and Rohrlich and described in chapter 1 of the book by S. Lang on «Complex multiplication», Springer, Berlin (1983).

REFERENCES

[1] P.J. RICHENS, M.V. BERRY: *Pseudo integrable systems in classical and quantum mechanics,* Physica **2D**, 495, 1981.

[2] L. GUILLOPÉ: *Billiards rationnels et espaces de Teichmüller* in *Séminaire de théorie spectrale et géométrie,* Chambery-Grenoble, 1986-1987, pages 33 et 42.

[3] C. ITZYKSON, J.M. LUCK, P. MOUSSA: *Sum rules for quantum billiards,* J. Phys. **A19**, L111, 1986.

[4] M. GAUDIN: *Reduction du problème du billiard quantique triangulaire,* Journal de Physique **48**, 1633, 1987.

[5] H.S.M. COXETER: *Regular Polytopes,* Dover (New York), 1973.

[6] C. ITZYKSON: *Simple integrable systems and Lie algebras,* International Journal of Modern Physics A, **1**, 1986.

Manuscript received: August 25, 1988